EMROZ'S

Calculus

100% Error-free & Self-sufficient

Why this Book?

In order to understand Physics of the present age one needs to master Calculus. To introduce Calculus, one may say that it is just like another mathematical tool such as Vector, Matrix, Sets, Tensors etc. which help us to carry out calculations and other tedious derivations associated with science much more easily. Therefore, such tools have received so much acceptance that almost every book relating to science, especially physics, has Calculus used in it indiscriminately. Therefore, any thinker, who is starting to be amused by the Physical laws governing the universe, must read Calculus beforehand; even if he is a high school student. If he does not do so, (in other words, if the school courses do not include calculus early) then he will be deprived of the total picture of physics and will be forced to obey the laws and theorems blindly, rather than examine it.

Such handicap in our country has been destroying the enthusiasm of the young learners towards science as well as their confidence to think something new, something theoretically innovative. Though, many persistent students opt to learn calculus by themselves by getting a textbook of Calculus, but soon they give up because of a new problem. The problem is illustrated as follows—

In describing velocities, physics says— " $v = \frac{dx}{dt}$ ". Furthermore, it says if " $x = t^3$, then, $v = \frac{d}{dt}(t^3) = 3t^2$ " Now, the persistent youngster wonders how $\frac{d}{dt}(t^3)$ got equal to $3t^2$. So, he gets to his Calculus textbook but discovers that the proof of this is established on another theorem called The Binomial theorem. This is not a part of Calculus; instead this theorem is proved in the algebra textbooks. Even if he reads the proof in Algebra, he will be disappointed again to find out that that proof is not self-sufficient either as it requires some concepts of Permutation & Combinations. Thus, the student is presented with a chain of difficulties and hardly manages absorbing the chain in the little time available.

Apart from The Binomial theorem, there are still more chains to deal with; such as— in the Simple Harmonic Motion, physics uses the formula— $\frac{d}{dx}(\sin x) = \cos x$, proof of which is based on advanced trigonometrical formulas. And these formulas are again based on some Geometry. Assessing the vastness of these chains, no teacher volunteers to teach Calculus early and all students prefer to be ignorant.

But I did not like the idea. I covered all the chains and found out an interesting discovery. I noticed that to understand Physics covered by our syllabus, a student need some specific ideas of Calculus. He does not have to go through all the pages of a Calculus textbook, rather some definite proofs; even though as a beginner I had to explore all the pages. The same thing is true for Trigonometry, The Binomial theorem and Permutations & Combinations. Thus armed with this idea of shortening the chains considerably and also being familiar with what a high school student has learnt so far, I decided to share my experience in a form of a booklet.

In this book, I intend to present every inquiring mind the chains coiled in one place. Although the book is named 'Calculus', nearly half of it deals with non-Calculus topics. Lastly, I strongly guarantee that after going through this book, any high school student will be adapted to understand physics.

EMROZ
June, 2007
Dhaka

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—To those—
who did not worship math
rather
created it in need
and
handled it like a tool
to bring out the truth

—1—
PERMUTATIONS & COMBINATIONS

The *collections* (ignoring the serial of its components) which can be made by taking the letters *a, b, c, d* two at a time are six in number, namely :

ab, ac, ad, bc, bd, cd

Each of the different collections above is called a combination.

The *arrangements* (minding the serial of its components) which can be made by taking the letters *a, b, c, d* two at a time are twelve in number, namely :

ab, ac, ad, bc, bd, cd
ba, ca, da, cb, db, dc

Each of the different arrangements above is called a permutation.

From this, it appears that in forming combinations, we are only concerned with the varieties of things each collection contains; whereas in forming permutations we also have to consider the order of the things which make up each arrangement. For instance, if from *a, b, c, d* we make a collection of three letters, such as *abc*, this single combination admits of being arranged in the following ways:

abc, acb, bca, bac, cab, cba

And so gives rise to six different permutations.

Here is an important principle which we proceed to explain below:

"If one operation can be performed in m *ways and (when it has been performed in any one of these ways) a second operation can then be performed in* n *ways; the number of ways of performing the two operations jointly will be* $m \times n$*."*

If the first operation is performed in *any one* of the m ways, we can associate with this any of the n ways by which the second operation can be performed. And thus we shall have n ways of performing the two operations jointly without considering more than *one* way of performing the first. And so, corresponding to *each* of the m ways of performing the first operation, we shall have n more ways of performing the two. Hence altogether, the number of ways in which the two operations can be performed is represented by the product $m \times n$.

Example: There are 10 trains running between Dhaka to Chittagong. In how many ways can a man go from Dhaka to Chittagong and return by a different train?

Solution: There are ten ways of making the first passage, and with each of these there is a choice of nine ways of returning (since the man is not to come back by the same train); hence, the number of ways of making the two journeys is $10 \times 9 = 90$.

The principle may easily be extended to the case in which there are more than two operations each of which can be performed in a given number of ways.

To find the number of permutations of n things taken r at a time:

This is the same thing as finding the number of ways in which we can fill up r places when we have n different things at our disposal.

The first place may be filled up in n ways, for any one of the n things may be taken; when it has been filled up in any one of these ways, the second place can then be filled up in n -1 ways; and since each way of filling up the first place can be associated with each way of filling up the second, the number of ways in which the first two places can be filled up is given by the product $n(n-1)$. And when the first two places have been filled up in any way, the third place can be filled up in n -2 ways. And reasoning as before the number of ways in which three places can be filled up is $n(n-1)(n-2)$.

Proceeding thus, and noticing that a new factor is introduced with each new place filled up, and that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up equal to—

$$n(n-1)(n-2)\ldots\ldots \text{ to } r \text{ factors} \quad \textit{or,} \quad (n-0)(n-1)(n-2)\ldots\ldots \text{ to } r \text{ factors}$$

And the r^{th} factor is $n-(r-1) = n-r+1$. Therefore, the number of permutations of n things taken r at a time is—

$$n(n-1)(n-2)\ldots\ldots(n-r+1);$$

which is denoted by the symbol ${}^nP_r$.

<u>Corollary:</u> The number of permutations of n things taken all at a time is—

$$n(n-1)(n-2)\ldots\ldots \text{ to } n \text{ factors} \quad \textit{or,} \quad n(n-1)(n-2)\ldots\ldots 3.2.1$$

It is usual to denote this product by the symbol $n!$ which is read 'factorial n'.

$\therefore {}^nP_n = n!$

<u>Example:</u> In an English exam, a rearrangement of ten sentences is given. In how many ways a student may answer it so that he fails to get full marks?

<u>Solution:</u> The ten sentences can be rearranged among themselves in ${}^{10}P_{10} = 10!$ ways. But there is only one correct order of them; which leaves $10!-1$ orders as incorrect sets of answers.

To find the number of combinations of n things taken r at a time:

Let ${}^nC_r$ denote the required number of combinations.

Then, each of these combinations consists of a collection of r things which can be arranged among themselves in $r!$ ways. Hence ${}^nC_r \times r!$ is equal to the number of permutations of n things taken r at a time; that is—

$^{n}C_{r} \times r! = {}^{n}P_{r}$

$= n\ (n-1)(n-2)\ldots\ldots\ (n-r+1)$

$\therefore\ {}^{n}C_{r} = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}$ ……… ………… (i)

$= \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)\times(n-r)!}{r!(n-r)!}$

$\therefore\ {}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ …………………(ii)

It is advisable to remember both the expression in (i) & (ii) for future use.

Corollary:

In making all the possible combinations of *n* things taken *r* at a time, to each group of *r* things we select, there is left a corresponding group of *n-r* things; that is, the number of combinations of *n* things *r* at a time is the same as the number of combinations of *n* things *n-r* at a time—

$$\therefore {}^{n}C_{n-r} = {}^{n}C_{r}$$

This can also be proved mathematically—

$${}^{n}C_{n-r} = \frac{n!}{(n-r)!\{n-(n-r)\}!} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{(n-r)!r!} = {}^{n}C_{r}$$

Example: In Bangladesh Cricket team, there are about 14 players. In how many ways a team of 11 players can be formed for a match against Australia?

Solution: A team is a gathering of players, regardless of their order. Hence, the required number of selecting players is— ${}^{14}C_{11} = {}^{14}C_{3} = \dfrac{14.13.12}{3!} = 364$

—2—
THEOREMS OF EXPANSIONS

- To expand $(a+x)^n$ where *n* is a positive whole number:

It may be shown by actual multiplication that—

$$(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 + (abc+abd+acd+bcd)x + abcd$$

We may however write down this result by using a different approach: for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, one being taken from each of the four factors. If we examine the way in which the various partial products are formed, we see that—

1) The term x^4 is formed by taking the letter *x* out of *each* of the factors.
2) The terms involving x^3 are formed by taking the letter *x* out of *any three* factors, in every way possible, and *one* of the letters *a*, *b*, *c*, *d* out of the remaining factor.
3) The terms involving x^2 are formed by taking the letter *x* out of *any two* factors, in every way possible, and *two* of the letters *a*, *b*, *c*, *d* out of the remaining factors.
4) The terms involving x are formed by taking the letter *x* out of *any one* factors, in every way possible, and *three* of the letters *a*, *b*, *c*, *d* out of the remaining factors.
5) The term independent of *x* is the product of all the letters *a*, *b*, *c*, *d*.

Applying similar approach we may write—

$$(x+a)(x+b)(x+c)......(x+k) = x^n + S_1x^{n-1} + S_2x^{n-2} + ... + S_rx^{n-r} + ... + S_{n-1}x + S_n$$

—where the number of factors on the left is *n* and

$$S_1 = a+b+c+......+k\,;$$
$$S_2 = ab+ac+...+ak+bc+bd+.......;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n = abc......k$$

Now, in S_1 the *number of terms* is *n*. In S_2 the *number of terms* is the same as the number of ways of selecting any two letters out of the *n* letters— *a, b, c, ..., k*. In other words, it is same as the number of combination of *n* things taken 2 at a time— that is, ${}^nC_2$. Similarly, in S_3 the number of terms is ${}^nC_3$ and so on.

Now, suppose *b, c, ..., k* all equal to *a*. Then, S_1 becomes *na* or ${}^nC_1a$, S_2 becomes ${}^nC_2a^2$, S_3 becomes ${}^nC_3a^3$ and so on until S_n becomes a^n. Thus we have—

$$(x+a)^n = x^n + {}^nC_1ax^{n-1} + {}^nC_2a^2x^{n-2} + ... + {}^nC_ra^rx^{n-r} + ... + a^n;$$

If we put $x=1$ & $a=x$, the expansion becomes——

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_rx^r + ... + x^n$$

The expansion stated above is known as the Binomial Theorem.

- To expand a^x in ascending powers of x:

By the Binomial Theorem we obtain——

$$\left(1+\frac{1}{n}\right)^{nx} = 1+{}^{nx}C_1\frac{1}{n}+{}^{nx}C_2\left(\frac{1}{n}\right)^2+{}^{nx}C_3\left(\frac{1}{n}\right)^3+\ldots\ldots\ldots\ldots\ldots+\left(\frac{1}{n}\right)^{nx}$$

$$=1+nx.\frac{1}{n}+\frac{nx(nx-1)}{2!}.\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{3!}.\frac{1}{n^3}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^{nx}}$$

$$=1+x+\frac{\frac{nx}{n}.\frac{(nx-1)}{n}}{2!}+\frac{\frac{nx}{n}.\frac{(nx-1)}{n}.\frac{(nx-2)}{n}}{3!}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^{nx}}$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{2!}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^{nx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

By putting $x=1$, we obtain——

$$\left(1+\frac{1}{n}\right)^{n} = 1+1+\frac{1-\frac{1}{n}}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^{n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

But,

$$\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}$$

Hence, the series (i) is the x^{th} power of the series (ii), that is——

$$1+x+\frac{x\left(x-\frac{1}{n}\right)}{2!}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^{nx}}$$

$$=\left\{1+1+\frac{1-\frac{1}{n}}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\ldots\ldots\ldots\ldots+\frac{1}{n^{n}}\right\}^{x}$$

And this is true however great *n* may be. If therefore *n* is indefinitely increased we have——

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\infty = \left(1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\ldots\infty\right)^{x}$$

The series $1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\ldots\infty$ is usually denoted by the symbol *e*; hence——

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\dots \infty$$

If we write cx for x, then——

$$e^{cx} = 1 + cx + \frac{c^2x^2}{2!} + \frac{c^3x^3}{3!} + \dots\dots\dots\dots \infty$$

Now, let $e^c = a$; so that $c = \log_e a = \ln a$.

$$\therefore a^x = 1 + x\ln a + \frac{x^2(\ln a)^2}{2!} + \frac{x^3(\ln a)^3}{3!} + \dots\dots\dots\dots \infty;$$

This is the Exponential Theorem.

To expand $\ln(1+x)$ in ascending powers of x:

By the Exponential Theorem we get——

$$a^y = 1 + y\ln a + \frac{y^2(\ln a)^2}{2!} + \frac{y^3(\ln a)^3}{3!} + \dots\dots\dots\dots \infty$$

Writing $1+x$ for a we get——

$$(1+x)^y = 1 + y\ln(1+x) + \frac{y^2\{\ln(1+x)\}^2}{2!} + \frac{y^3\{\ln(1+x)\}^3}{3!} + \dots\dots\dots\dots \infty \qquad \dots\dots\dots\dots\dots\text{(i)}$$

But, by the Binomial theorem we get——

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{2!}x^2 + \frac{y(y-1)(y-2)}{3!}x^3 + \frac{y(y-1)(y-2)(y-3)}{4!}x^4 + \dots\dots + x^y$$

$$= 1 + yx + \frac{y^2 - y}{2!}x^2 + \frac{y^3 - 3y^2 + 2y}{3!}x^3 + \frac{y^4 - 6y^3 + 11y^2 - 6y}{4!}x^4 + \dots\dots + x^y$$

$$= 1 + yx + \frac{y^2}{2!}x^2 - y.\frac{x^2}{2!} + \frac{y^3 - 3y^2}{3!}x^3 + y\frac{2x^3}{3!} + \frac{y^4 - 6y^3 + 11y^2}{4!}x^4 - y\frac{6x^4}{4!} + \dots\dots + x^y$$

$$= 1 + y\left(x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots\right) + \frac{y^2}{2!}x^2 + \frac{y^3 - 3y^2}{3!}x^3 + \frac{y^4 - 6y^3 + 11y^2}{4!}x^4 + \dots\dots + x^y$$

$$\dots\dots\dots\dots\dots\dots\text{(ii)}$$

Now, (i) & (ii) are equal. So, the coefficients of y of the same degree in both the series must be equal. Hence——

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots\dots \text{[Coefficients of } y \text{ of the first degree in the two series]}$$

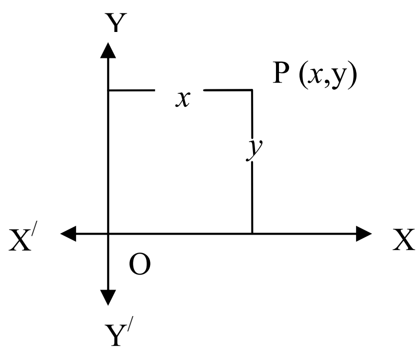
$$= x - \frac{x^2}{2\times 1} + \frac{2x^3}{3\times 2\times 1} - \frac{6x^4}{4\times 3\times 2\times 1} + \dots\dots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots;$$

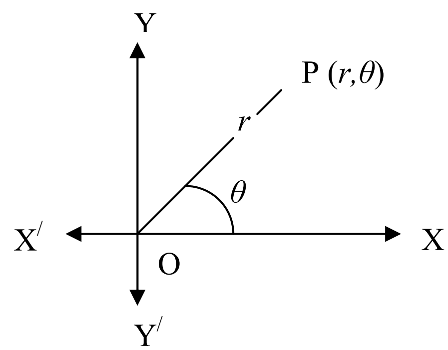
This is known as the Logarithmic Series.

—3—
TRIGONOMETRY

- In graphs we can locate any point by either of the following two methods:
  1) By measuring distances from the two axis X and Y.
  2) By measuring the distance from the Origin point and the angle with the X-axis.

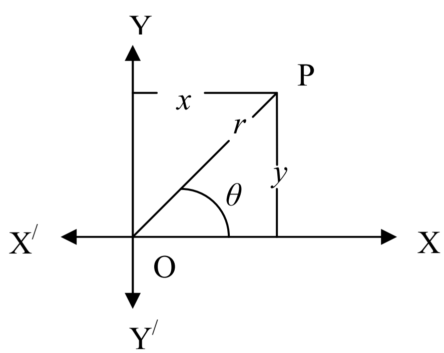


(1) Rectangular Coordinate



(2) Polar Coordinate

Now, let's find out a relationship by which we may switch a coordinate from Polar to Rectangular system.



Let P (r, θ) be a point. Then, according to the figure above, we get——

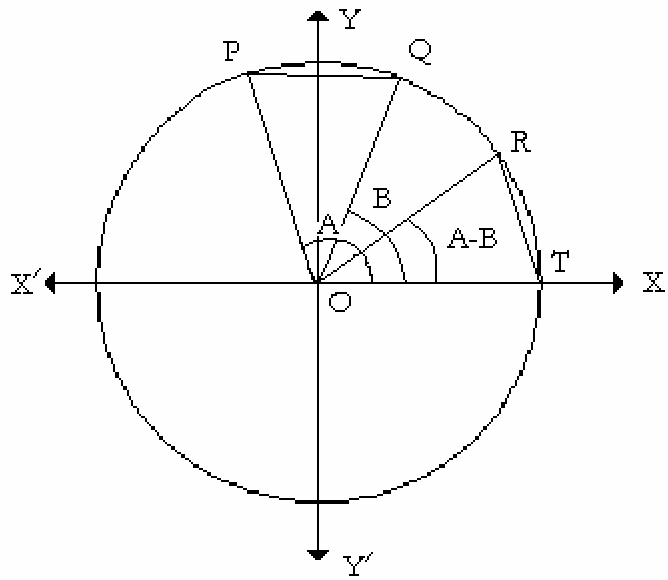
$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \qquad \& \qquad \sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

Then, the rectangular coordinate of P will be (x,y) = $(r\cos\theta, r\sin\theta)$.

<u>Corollary:</u> If P is (1, θ) in the Polar system, then in the rectangular one it is ($\cos\theta$, $\sin\theta$).

- Let A, B & A-B be angles of any value. In graphs, let's suppose——

$$\angle XOP = A,\ \ \angle XOQ = B \text{ and } \angle XOR = A - B.$$



Now, a circle of unit radius centering the Origin point intersects OP, OQ, OR & the X-axis at the points P, Q, R & T respectively. Then, the polar coordinates of those points will be (1, A), (1, B), (1, A-B) & (1,0). Transforming them into rectangular coordinates, we obtain—

P (cosA, sinA), Q (cosB, sinB), R (cos(A-B), sin(A-B)) & T (1,0).

As, $\angle ROT = A - B = \angle POT - \angle QOT = \angle POQ$

So, $RT = PQ$ ……………(i)

But we know, in graphs distance between the two points $\mathrm{M}(x_1, y_1)$ and $\mathrm{N}(x_2, y_2)$ is——

$$\mathrm{MN} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Then, from (i) we obtain——

$$\sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$\Rightarrow \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(\mathrm{A - B})$$
$$= \cos^2\mathrm{A} - 2\mathrm{cosA.cosB} + \cos^2\mathrm{B} \quad + \quad \sin^2 A - 2\sin A.\sin B + \sin^2 B$$

$$\Rightarrow \{\cos^2(A-B) + \sin^2(A-B)\} + 1 - 2\cos(A-B)$$
$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A.\cos B + \sin A.\sin B)$$

$$\Rightarrow 1 + 1 - 2\cos(A-B) = 1 + 1 - 2(\cos A.\cos B + \sin A.\sin B)$$

$$\therefore \cos(A-B) = \cos A.\cos B + \sin A.\sin B \text{ ....(i)}$$

Now, writing $(-B)$ for B we get——

$\cos\{A-(-B)\} = \cos A.\cos(-B) + \sin A.\sin(-B)$

$\therefore \cos(A+B) = \cos A.\cos B - \sin A.\sin B$ ....(ii)

Again, squaring both sides of (i) we obtain——

$\cos^2(A-B) = \cos^2 A.\cos^2 B + 2\cos A.\cos B.\sin A.\sin B + \sin^2 A.\sin^2 B$

$\Rightarrow 1-\sin^2(A-B) = (1-\sin^2 A)\cos^2 B + 2\cos A.\cos B.\sin A.\sin B + (1-\cos^2 A)\sin^2 B$

$\Rightarrow 1-\sin^2(A-B)$

$= \cos^2 B - \sin^2 A.\cos^2 B + 2\cos A\cos B\sin A\sin B + \sin^2 B - \sin^2 B.\cos^2 A$

$\Rightarrow 1-\sin^2(A-B)$

$= (\cos^2 B + \sin^2 B) - (\sin^2 A.\cos^2 B - 2\sin A.\cos B.\sin B.\cos A + \sin^2 B.\cos^2 A)$

$\Rightarrow 1-\sin^2(A-B) = 1-(\sin A.\cos B - \sin B.\cos A)^2$

$\therefore \sin(A-B) = \sin A.\cos B - \sin B.\cos A$ ………………….(iii)

Now, writing $(-B)$ for B we get——

$\sin\{A-(-B)\} = \sin A.\cos(-B) - \sin(-B).\cos A$

$\therefore \sin(A+B) = \sin A.\cos B + \sin B.\cos A$ ………………….(iv)

Subtracting (iii) from (iv) we get——

$\sin(A+B) - \sin(A-B) = 2\sin B.\cos A$ …………………....(v)

Subtracting (i) from (ii) we get——

$\cos(A+B) - \cos(A-B) = -2\sin A.\sin B$ ……………….(vi)

Suppose, $A+B=C$ & $A-B=D$

Then, $A = \dfrac{C+D}{2}$ & $B = \dfrac{C-D}{2}$

So, (v) and (vi) becomes——

$$\sin C - \sin D = 2\sin\frac{C-D}{2}.\cos\frac{C+D}{2};$$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}$$

—4—
LIMITS

- Consider the following function:

$$f(x) = x + 2$$

Here, we may take any value of x as we please. For example, $f(2) = 4$. But——

$$f(x) = x + 2 = \frac{(x+2)(x-2)}{(x-2)} = \frac{x^2 - 4}{x - 2}$$

Now, $$f(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$$

So, there is a little restriction upon x at the moment since for its value as 2 the function becomes invalid. But, we have seen earlier that there really had been an actual value of $f(2)$. Therefore, Mathematics needs a new branch to deal with such problems. Hence, arises the idea of Limits.

For the previous function $f(x) = \frac{x^2 - 4}{x - 2}$, we can not put x =2 directly; but however, we are not prohibited from taking values of x *nearly* 2:

$$f(1.9) = 3.9 \qquad f(2.1) = 4.1$$
$$f(1.99) = 3.99 \qquad f(2.01) = 4.01$$
$$f(1.999) = 3.999 \qquad f(2.001) = 4.001$$

So, as we can see that the results are also *nearly* 4 —— the actual value of $f(2)$. Moreover, from the values above we can also see that—— the more nearly we take the value of x towards 2, the more accurate the result occurs. In other words, if the value we take for x differs from 2 by as little as possible (nearly 0), our achieved result will also differ from the actual one by nearly 0. Let, h be the difference. Then——

$$f(2+h) = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = \frac{h(4+h)}{h} = 4 + h \text{ ; nearly 4.}$$

Now, however small h may be, $f(2)$ will not be exactly 4 until h gets exactly equal to zero. We see above that putting $h = 0$ in the first part, gives us an invalid value $\frac{0}{0}$ which is neither zero nor infinity. But, putting $h = 0$ in the last part instead, does give us $f(2) = 4$.

Therefore, a slight modification of the function as well as its domain has enabled us to achieve a valid value (or historically called a *limiting value)*. And the process of modification is recognized as finding the limit of a function.

The method of modification employed here is usable for any function; however complex the function may be. As a result, such operations should be put into symbolic expressions:

In a function of x*, namely* f(x)*, if* x *approaches a value* a *(that is:* $x \to a$*), the function switches to a result nearly* R *(that is:* $f(x) \to R$*), then* R *is called the Limit of the given function when* x *tends to* a and *this is written by*——

$$\lim_{x \to a} f(x) = R$$

Then, according to our last example——

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

It should be noted that the concept of limit is the basis of Calculus. But, many students find it as a weak idea since it talks about approximations and approaches. But it is *not* the case. Limiting a function means to find a valid image for an assigned value of the variable; whether the assigned value is in the domain of the function or not. It is as though *breaking the limit of the domain*. Examples of approximations are given only to introduce limits easily. They are illustrated only to show that values near the assigned value surround and point towards the required valid image; even though the assigned value gives an invalid result directly.

In practical use, while $x \to 0$, our aim should be to try to take the given expression to a position where we can insert the negligible value of x and with that calculate the required answer. Moreover, in other cases, we may always introduce the difference indicating symbol h (as shown before) and proceed in the same manner. The following examples illustrate the procedure:

Example: Find out $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Solution: The question asks for the limit of $\frac{e^x - 1}{x}$ when x tends to 0; that is, a value to which $\frac{e^x - 1}{x}$ will approach when x approaches 0.

Clearly, we can not put x' s value now. So, let's try to remove x from the denominator.

$$\therefore \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\infty\right) - 1}{x}$$

$$= \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\infty}{x}$$

$$= \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\ldots\ldots\ldots\ldots\infty\right)$$

Now, in the expression above, we have no hesitation to put the given value of x.

$$\therefore \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\ldots\ldots\ldots\ldots\infty\right) = 1$$

$$\therefore \lim_{x\to 0}\frac{e^x-1}{x}=1$$

<u>Example:</u> Find out $\lim_{x\to a}\frac{x^n-a^n}{x-a}$. (That is: if $x\to a$, $\frac{x^n-a^n}{x-a}\to ?$)

<u>Solution:</u> Let, $x=a+h$ $\therefore h=x-a$; h may be positive or negative. Now,

$$\because x\to a$$
$$\therefore h\to 0$$

Then, $$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{h\to 0}\frac{(a+h)^n-a^n}{a+h-a} = \lim_{h\to 0}\frac{(a+h)^n-a^n}{h} = \lim_{h\to 0}\frac{a^n\left(1+\frac{h}{a}\right)^n-a^n}{h}$$

Applying the Binomial Theorem we continue——

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{h\to 0}\frac{a^n\left\{1+n.\frac{h}{a}+\frac{n(n-1)}{2!}.\frac{h^2}{a^2}+\frac{n(n-1)(n-2)}{3!}.\frac{h^3}{a^3}+\ldots\ldots\ldots\ldots\right\}-a^n}{h}$$

$$=\lim_{h\to 0}\frac{\left\{a^n+na^{n-1}h+\frac{n(n-1)}{2!}a^{n-2}h^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}h^3+\ldots\ldots\ldots\ldots\right\}-a^n}{h}$$

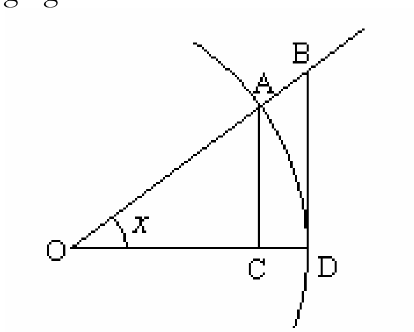
$$=\lim_{h\to 0}\frac{na^{n-1}h+\frac{n(n-1)}{2!}a^{n-2}h^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}h^3+\ldots\ldots\ldots\ldots}{h}$$

$$=\lim_{h\to 0}\left\{na^{n-1}+\frac{n(n-1)}{2!}a^{n-2}h+\frac{n(n-1)(n-2)}{3!}a^{n-3}h^2+\ldots\ldots\ldots\ldots\right\}$$

$= na^{n-1}$; this is the required answer.

- Fundamental properties of Limits:
  i. $\lim\limits_{x\to a}\{f(x)+\phi(x)\} = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a}\phi(x)$
  ii. $\lim\limits_{x\to a}\{f(x)-\phi(x)\} = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a}\phi(x)$
  iii. $\lim\limits_{x\to a}\{f(x)\times\phi(x)\} = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a}\phi(x)$
  iv. $\lim\limits_{x\to a}\dfrac{f(x)}{\phi(x)} = \lim\limits_{x\to a} f(x) \div \lim\limits_{x\to a}\phi(x)$; $\quad [\lim\limits_{x\to a}\phi(x) \neq 0]$

- Consider the following figure:



Here, $OA = OD,\ AC \perp OD,\ BD \perp OD$ and $\angle AOC = x$ radian $= x^c = x$

Now, the area of $\Delta OAC = \frac{1}{2}OC.AC = \frac{1}{2}.\frac{OC}{OA}.\frac{AC}{OA}.OA^2 = \frac{1}{2}OA^2.\cos x.\sin x$

We know, the area of a circle is πr^2 and the angle made to the centre is 2π. Then——

The area of the part of a circle making an angle 2π to the centre is πr^2

$\therefore$ The area of the part of a circle making an angle 1 to the centre is $\pi r^2/2\pi$

$\therefore$ The area of the part of a circle making an angle θ to the centre is $\pi r^2\theta/2\pi = \frac{1}{2}r^2\theta$

$\therefore$ The area of $OAD = \frac{1}{2}OA^2.x$

Again, the area of $\Delta OBD = \frac{1}{2}OD.BD = \frac{1}{2}OD^2.\frac{BD}{OD} = \frac{1}{2}OD^2.\tan x = \frac{1}{2}OA^2.\tan x$

Now, clearly from the figure——

The area of ΔOAC $<$ The area of OAD $<$ The area of ΔOBD

$$\therefore \frac{1}{2}OA^2.\cos x.\sin x < \frac{1}{2}OA^2.x < \frac{1}{2}OA^2.\tan x$$

$$\therefore \cos x.\sin x < x < \frac{\sin x}{\cos x}$$

$$\therefore \cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\therefore \frac{1}{\cos x} > \frac{\sin x}{x} > \cos x \qquad \text{[Inverting the fractions]}$$

Now, this inequality is true for any value of *x*. If $x \to 0$, then $CD \to 0$, in other words, $OC \to OD$, that is, $OC \to OA$ Hence——

$$\lim_{x\to 0}\frac{1}{\cos x} = \lim_{x\to 0}\sec x = \lim_{OC\to OA}\frac{OA}{OC} = 1 \qquad \& \qquad \lim_{x\to 0}\cos x = \lim_{OC\to OA}\frac{OC}{OA} = 1$$

So, if $x \to 0$, we get from the previous inequality——

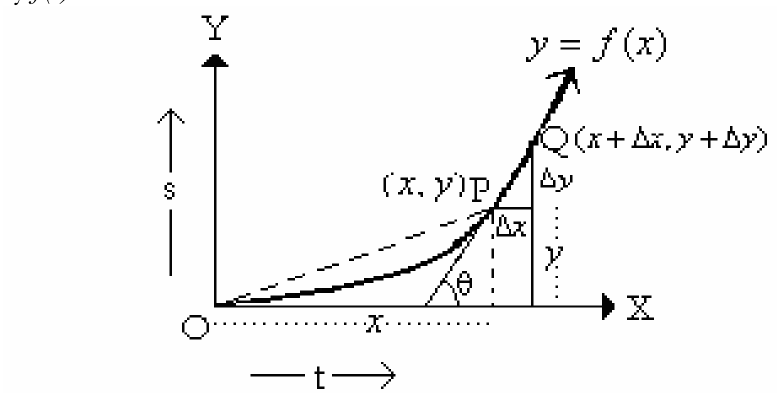
$$\lim_{x\to 0}\frac{1}{\cos x} > \lim_{x\to 0}\frac{\sin x}{x} > \lim_{x\to 0}\cos x$$

$$1 > \lim_{x\to 0}\frac{\sin x}{x} > 1;$$

which means: $$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

—5—
DIFFERENTIALS

- Let the next figure be a distance-time graph of a running vehicle. After a certain time, *t;* let's suppose that the vehicle's displacement, *s* follows a definite function of *t*, namely *f(t)*.



Symbolizing *t* in the X-axis and *s* in the Y-axis, we get a curve, $y = f(x)$. Then, let's determine the value of the vehicle's instant velocity at any point in the graph; P (*x,y*), for example.

Now, $\frac{y}{x}$ can not be the required velocity, since it is the average velocity of the straight line *OP*; which derails from our given function, $y = f(x)$ a lot. So, we have to think of another way. Let's take a point, Q (x+Δx, y+Δy), right next to P. Jointing PQ, we again get a straight line. Let's suppose it makes an angle, θ with the X-axis.

Now, if we keep dragging Q closer to P; that is, make Δx & Δy nearly zero; in other words, make the line PQ nearly a tangent on the curve, then P & Q will be so close that we can consider the values of velocity lying between them nearly constant. Hence, the average value of velocity of that little part of the whole curve will approximate to the instant value at the point P.

$\therefore$ The required instant value at P = The average velocity of PQ $= \frac{\Delta y}{\Delta x}$; but still it is not exactly the *instant* velocity unless Δx & Δy are exactly equal to zero. But again we see that putting Δx & Δy equal to zero gives us $\frac{\Delta y}{\Delta x} = \frac{0}{0}$; which is invalid. Therefore, to achieve a valid value for the velocity we recall to use limits.

$\therefore$ The required instant value $= \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta y}{\Delta x} = \tan\theta$

But, since y is a function of x, we get $\Delta y \to 0$ automatically when $\Delta x \to 0$. So, abbreviating the above expression we get $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Abbreviating more, we write $\frac{dy}{dx}$ and read 'dd$x$ of y'.

$$\therefore \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x};$$

Denoting Δx by h, we get——

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here, $\frac{d}{dx} f(x)$ is called the differential co-efficient or derivative of the function $f(x)$ and since it is a function, it may also be denoted by $f'(x)$. Now, $f'(m)$ indicates the rate of change of the function *f(x)* at the point (m, *f*(m)) in graph. To say more clearly——

- ✓ If m belongs to the domain, the curve of the function *f(x)* must go over the point (m, *f*(m)) in graph.
- ✓ From the point (m, *f(m)*), the curve will face a change since it may rise upward (positive change) or fall downward (negative change) or remain horizontal (null change).
- ✓ If there is a change, there must be a rate of change by which we may assume the amount of change or the amount of bending the curve.
- ✓ In our previous discussion, the instant velocity at any point will be 0, if the vehicle's displacement vs. time curve flows horizontally; will be increased if the curve runs above straight enough. So, instant velocity indicates the rate of change of the curve. But, we can know the instant velocity by a derivative. So, we can also know the rate of change.
- ✓ So, a derivative $f'(x)$ is a function which indicates the rate of change of *f(x)*. So, $f'(m)$ is a value which is the rate of change of *f(x)* at the point (m, *f*(m)).

Derivatives of some useful functions are deduced below——

- If, $f(x) = c$, where c is a constant; then——

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\therefore \frac{dc}{dx} = 0.$$

- $$\frac{d}{dx}(x^n) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{x^n\left\{1+n.\frac{h}{x}+\frac{n(n-1)}{2!}.\frac{h^2}{x^2}+\frac{n(n-1)(n-2)}{3!}.\frac{h^3}{x^3}+.........\right\}-x^n}{h}$$
$$= \lim_{h\to 0}\frac{\left\{x^n + nx^{n-1}h+\frac{n(n-1)}{2!}x^{n-2}h^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}h^3+...........\right\}-x^n}{h}$$
$$= \lim_{h\to 0}\frac{nx^{n-1}h+\frac{n(n-1)}{2!}x^{n-2}h^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}h^3+...........}{h}$$
$$= \lim_{h\to 0}\left\{nx^{n-1}+\frac{n(n-1)}{2!}x^{n-2}h+\frac{n(n-1)(n-2)}{3!}x^{n-3}h^2+..........\right\}$$
$$= nx^{n-1}$$
$$\therefore \frac{d}{dx}(x^n) = nx^{n-1}$$

Corollary: $\frac{dx}{dx} = \frac{d}{dx}x^1 = 1.x^{1-1} = x^0 = 1$

- $$\frac{d}{dx}(e^{mx}) = \lim_{h\to 0}\frac{e^{m(x+h)} - e^{mx}}{h}$$
$$= \lim_{h\to 0}\frac{e^{mx}(e^{mh}-1)}{h}$$
$$= \lim_{h\to 0}\frac{e^{mx}\left\{\left(1+mh+\frac{m^2h^2}{2!}+\frac{m^3h^3}{3!}+.................\right)-1\right\}}{h}$$
$$= \lim_{h\to 0}\frac{e^{mx}}{h}\left(mh+\frac{m^2h^2}{2!}+\frac{m^3h^3}{3!}+.................\right)$$
$$= \lim_{h\to 0}\left(me^{mx}+\frac{m^2h}{2!}e^{mx}+\frac{m^3h^2}{3!}e^{mx}+.................\right)$$
$$= me^{mx}$$

$$\therefore \frac{d}{dx}(e^{mx}) = me^{mx}$$

Corollary: If, $m = 1$, then the above equation stands: $\frac{d}{dx}(e^x) = e^x$

➢ $$\frac{d}{dx}(\ln x) = \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h}$$
$$= \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{x} - \frac{1}{2}.\frac{h^2}{x^2} + \frac{1}{3}.\frac{h^3}{x^3} - ....\right)$$ [From the Logarithmic series]
$$= \lim_{h\to 0}\left(\frac{1}{x} - \frac{1}{2}.\frac{h}{x^2} + \frac{1}{3}.\frac{h^2}{x^3} - ....\right)$$
$$= \frac{1}{x}$$
$$\therefore \frac{d}{dx}(\ln x) = \frac{1}{x}$$

➢ $$\frac{d}{dx}(\sin ax) = \lim_{h\to 0}\frac{\sin a(x+h) - \sin ax}{h}$$
$$= \lim_{h\to 0}\frac{\sin(ax+ah) - \sin ax}{h}$$
$$= \lim_{h\to 0}\frac{2\sin\frac{ax+ah-ax}{2}.\cos\frac{ax+ah+ax}{2}}{h}$$ $\left[\begin{array}{l}\because \sin C - \sin D \\ = 2\sin\frac{C-D}{2}.\cos\frac{C+D}{2}\end{array}\right]$
$$= \lim_{h\to 0}\frac{\sin\frac{ah}{2}.\cos\left(ax+\frac{ah}{2}\right)}{\frac{h}{2}}$$

$$= \lim_{h\to 0}\frac{\sin\frac{ah}{2}.a\cos\left(ax+\frac{ah}{2}\right)}{\frac{ah}{2}} = \lim_{h\to 0}\frac{\sin\frac{ah}{2}}{\frac{ah}{2}} \times \lim_{h\to 0} a\cos\left(ax+\frac{ah}{2}\right) = 1\times a\cos ax = a\cos ax.$$

$$\therefore \frac{d}{dx}(\sin ax) = a\cos ax$$

<u>Corollary:</u> If $a=1,$ then the above equation stands: $\frac{d}{dx}(\sin x) = \cos x$

➤ $$\frac{d}{dx}(\cos ax) = \lim_{h\to 0}\frac{\cos a(x+h)-\cos ax}{h}$$

$$= \lim_{h\to 0}\frac{\cos(ax+ah)-\cos ax}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\frac{ax+ah+ax}{2}.\sin\frac{ax+ah-ax}{2}}{h}$$

$$\left[\because \cos C-\cos D = -2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}\right]$$

$$= \lim_{h\to 0}\frac{-\sin\left(ax+\frac{ah}{2}\right).\sin\frac{ah}{2}}{\frac{h}{2}}$$

$$= \lim_{h\to 0}\frac{-a\sin\left(ax+\frac{ah}{2}\right).\sin\frac{ah}{2}}{\frac{ah}{2}}$$

$$= \lim_{h\to 0}\frac{\sin\frac{ah}{2}}{\frac{ah}{2}} \times \lim_{h\to 0}\left\{-a\sin\left(ax+\frac{ah}{2}\right)\right\} = 1\times(-a\sin ax) = -a\sin ax$$

$$\therefore \frac{d}{dx}(\cos ax) = -a\sin ax$$

<u>Corollary:</u> If $a=1,$ then the above equation stands: $\frac{d}{dx}(\cos x) = -\sin x$

➢ $$\frac{d}{dx}(\tan x) = \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{\sin(x+h)}{\cos(x+h)} - \dfrac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h).\cos x - \sin x.\cos(x+h)}{h\cos(x+h).\cos x}$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h).\cos x} \qquad [\because \sin(A-B) = \sin A.\cos B - \sin B.\cos A]$$

$$= \lim_{h\to 0}\frac{\sinh}{h} \times \lim_{h\to 0}\frac{1}{\cos(x+h).\cos x}$$

$$= 1\times\frac{1}{\cos x.\cos x}$$

$$= \sec^2 x$$

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$$

Let u and v both be functions of x. If $y = u \pm v$, then y is another function of x. Now, if x undergoes a change Δx, then there will be corresponding changes in y, u & v; namely Δy, Δu & Δv. Hence, from $y = u \pm v$, we get——

$$y + \Delta y = (u+\Delta u) \pm (v + \Delta v) = u + \Delta u \pm v \pm \Delta v = (u \pm v) + (\Delta u \pm \Delta v) = y + (\Delta u \pm \Delta v)$$

$$\therefore \Delta y = \Delta u \pm \Delta v$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x}$$

$$\therefore \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} \pm \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\therefore \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

This formula may easily be extended to the case where there are more than two functions.

Let u and v both be functions of x. If $y = uv$, then y is another function of x. Now, if x undergoes a change Δx, then there will be corresponding changes in y, u & v; namely Δy, Δu & Δv. Hence, from $y = uv$, we get——

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\Delta v + v\Delta u + (\Delta u)(\Delta v) = y + u\Delta v + v\Delta u + (\Delta u)(\Delta v)$$

$$\therefore \Delta y = u\Delta v + v\Delta u + (\Delta u)(\Delta v)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{u\Delta v}{\Delta x} + \frac{v\Delta u}{\Delta x} + \frac{(\Delta u)(\Delta v)}{\Delta x}$$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{u\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \frac{v\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{(\Delta u)(\Delta v)}{\Delta x}$$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u \times \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \quad + \quad v \times \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \quad + \quad \lim_{\Delta x \to 0} \Delta u \times \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

But, if $\Delta x \to 0$, then $\Delta u \to 0$. Therefore, $\lim_{\Delta x \to 0} \Delta u = 0$.

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u \times \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \quad + \quad v \times \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \quad + \quad 0 \times \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

This formula may also be extended to the case where there are more than two functions.

Corollary: If, a is a constant and u is a function of x, then——

$$\frac{d}{dx}(au) = a\frac{du}{dx} + u\frac{da}{dx}$$ [Since, a can be considered as a function; such as, $f(x) = 0.x + a$.]

$$= a\frac{du}{dx} + u \times 0 \quad = \quad a\frac{du}{dx}.$$

$$\therefore \frac{d}{dx}(au) = a\frac{du}{dx}$$

Example: Find out $\frac{d}{dx}\left(\frac{x^{11} - 9x^5 + 3x}{x^8}\right)$.

Solution: $\frac{d}{dx}\left(\frac{x^{11} - 9x^5 + 3x}{x^8}\right) = \frac{d}{dx}(x^3 - 9x^{-3} + 3x^{-7}) = \frac{d}{dx}(x^3) - \frac{d}{dx}(9x^{-3}) + \frac{d}{dx}(3x^{-7})$

$$= 3x^2 - 9(-3)x^{-4} + 3(-7)x^{-8}$$

$$= 3x^2 + 27x^{-4} - 21x^{-8};$$

——this is the required derivative.

Let, y be a function of z (for example: $y=\sqrt{z}$) and z be a function of x (for example: $z=x+5$). Now, if x undergoes a change Δx, then there will be a corresponding change Δz, in the function z. Hence, for the same reason, there will also be a corresponding change Δy, in the function y (as y can be written as $y=\sqrt{x+5}$). Now by normal algebra we may write——

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta z}\times\frac{\Delta z}{\Delta x}$$

$$\therefore \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta z}\times\frac{\Delta z}{\Delta x}\right)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta z}\times\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}$$

Since, z is a function of x, therefore, if $\Delta x\to 0$, then, $\Delta z\to 0$. So——

$$\therefore \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta z\to 0}\frac{\Delta y}{\Delta z}\times\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$

This formula (known as the Chain rule) may also be extended to the case where there are more than two functions.

Let, y be a function of x. Then, if x undergoes a change Δx, then there will be a corresponding change Δy, in the function y. Now, by normal algebra we may write—

$$\frac{\Delta y}{\Delta x}=\frac{1}{\frac{\Delta x}{\Delta y}}$$

$$\therefore \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\frac{\Delta x}{\Delta y}}=\frac{1}{\lim\limits_{\Delta x\to 0}\frac{\Delta x}{\Delta y}}$$

Since, y is a function of x, therefore, if $\Delta x\to 0$, then, $\Delta y\to 0$. So——

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{1}{\lim\limits_{\Delta y\to 0}\frac{\Delta x}{\Delta y}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

<u>Example:</u> Find out $\frac{d}{dx}(\sin^2 3x)$.

<u>Solution:</u> $\frac{d}{dx}(\sin^2 3x) = \frac{d}{dx}\{(\sin 3x)^2\} = \frac{d}{d(\sin 3x)}\{(\sin 3x)^2\} \times \frac{d}{d(3x)}(\sin 3x) \times \frac{d}{dx}(3x)$

$$= 2\sin 3x \times \cos 3x \times 3.1$$
$$= 6\sin 3x.\cos 3x$$

<u>Example:</u> Find out $\frac{d}{dx}(\sin^{-1} x)$.

<u>Solution:</u> Let, $\theta = \sin^{-1} x$. Then, $\sin\theta = x$. Now,

$$\frac{d}{dx}(\sin^{-1} x) = \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{\frac{d}{d\theta}(\sin\theta)} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-(\sin\theta)^2}} = \frac{1}{\sqrt{1-x^2}}.$$

According to our last example, we could determine instant velocities from a displacement vs. time graph by deducing a derivative of the function *f(x)*, namely $\frac{d}{dx}f(x)$ or $f'(x)$. Furthermore, if we are asked to determine the instant accelerations, we may draw a velocity vs. time graph by following $y = \frac{d}{dx}f(x)$. Then, taking velocity as displacement, we can again deduce another derivative, namely $\frac{d}{dx}\left\{\frac{d}{dx}f(x)\right\}$ or $f''(x)$, which will indicate the rate of change of velocities; that is instant accelerations; just like $\frac{d}{dx}f(x)$ indicated the rate of change of displacements; that is instant velocities, before. Here, $\frac{d}{dx}\left\{\frac{d}{dx}f(x)\right\}$ may be written as $\frac{d^2}{dx^2}f(x)$ symbolizing the second derivative of the function $f(x)$. In extension, we may introduce $\frac{d^n}{dx^n}f(x)$ as the n^{th} derivative of $f(x)$.

—6—
INTEGRALS

In mathematics, for any operator there is always an inverse one, such as 'log & anti-log', 'sin & $\sin^{-1}$', '^{2} & $\sqrt{\ }$' etc. As a result, there is also an inverse operation of Differentiation, known as Integration. Symbolically, if $\frac{d}{dx}f(x) = f'(x)$, then—— $f(x) = \int f'(x)dx$. Here, dx stands for 'integration with respect to x'. Let c be a constant.

$$\therefore \frac{d}{dx}\{f(x)+c\} = \frac{d}{dx}f(x) + \frac{dc}{dx} = f'(x) + 0 = f'(x)$$

$$\therefore f(x) + c = \int f'(x)dx$$

So, we see that, $\int 3x^2 dx$ can be x^3 or $x^3 + 5$ or $x^3 - 2$. For this, it is wise to write——

$$\int 3x^2 dx = x^3 + c$$

However, we shall omit c in future to avoid unnecessary complication.

Let, $F_1(x) = \int f_1(x)dx$ & $F_2(x) = \int f_2(x)dx$, then——

$$\therefore \frac{d}{dx}\{F_1(x) \pm F_2(x)\} = \frac{d}{dx}F_1(x) \pm \frac{d}{dx}F_2(x) = f_1(x) \pm f_2(x)$$

Hence, according to the definition of integration, we obtain——

$$\int \{f_1(x) \pm f_2(x)\}dx = F_1(x) \pm F_2(x) = \int f_1(x)dx \pm \int f_2(x)dx$$

So, generally——

$$\int \{f_1(x) \pm f_2(x) \pm ..... \pm f_n(x)\}dx = \int f_1(x)dx \pm \int f_2(x)dx \pm .... \pm \int f_n(x)dx$$

Let, $F(x) = \int f(x)dx$, hence $\frac{d}{dx}F(x) = f(x)$. If a is any constant, then——

$$\frac{d}{dx}\{aF(x)\} = a\frac{d}{dx}F(x) = af(x)$$

$$\therefore \int af(x)dx = aF(x) = a\int f(x)dx$$

$$\therefore \int af(x)dx = a\int f(x)dx$$

- Integrations of some useful functions are deduced below:

- To determine $\int x^n dx$: $(n \neq -1)$

We know: $$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}.\frac{d}{dx}(x^{n+1}) = \frac{1}{n+1}(n+1)x^{(n+1)-1} = x^n$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1}$$

<u>Corollary:</u> $\int dx = \int 1.dx = \int x^0.dx = \frac{x^1}{1} = x$

- To determine $\int \frac{1}{x}dx$: Since, $\frac{d}{dx}(\ln x) = \frac{1}{x}$; hence, $\int \frac{1}{x}dx = \ln x$

- To determine $\int \ln x \; dx$:

Since, $\frac{d}{dx}(x\ln x - x)$

$$= \frac{d}{dx}(x\ln x) - \frac{dx}{dx} = \left\{x\frac{d}{dx}(\ln x) + \ln x.\frac{dx}{dx}\right\} - 1 = \left(x.\frac{1}{x} + \ln x.1\right) - 1 = 1 + \ln x - 1 = \ln x$$

$$\therefore \int \ln x \; dx = x\ln x - x$$

- To determine $\int \sin ax \; dx$:

Since,

$$\frac{d}{dx}(-\frac{1}{a}\cos ax) = -\frac{1}{a}.\frac{d}{dx}(\cos ax) = -\frac{1}{a}.\frac{d}{d(ax)}(\cos ax)\times\frac{d}{dx}(ax) = -\frac{1}{a}(-\sin ax).a.1 = \sin ax$$

$$\therefore \int \sin ax \; dx = -\frac{1}{a}\cos ax$$

<u>Corollary:</u> Putting $a = 1$, the above equation stands: $\int \sin x \; dx = -\cos x$

- To determine $\int \cos ax \; dx$:

Since,

$$\frac{d}{dx}(\frac{1}{a}\sin ax) = \frac{1}{a}.\frac{d}{dx}(\sin ax) = \frac{1}{a}.\frac{d}{d(ax)}(\sin ax)\times\frac{d}{dx}(ax) = \frac{1}{a}.\cos ax \times a.1 = \cos ax$$

$$\therefore \int \cos ax \; dx = \frac{1}{a}\sin ax$$

<u>Corollary:</u> Putting $a = 1$, the above equation stands: $\int \cos x \; dx = \sin x$

➢ To determine $\int e^{mx}dx$: Since, $\frac{d}{dx}\left(\frac{1}{m}e^{mx}\right) = \frac{1}{m}.\frac{d}{dx}(e^{mx}) = \frac{1}{m}.me^{mx} = e^{mx}$

$$\therefore \int e^{mx}dx = \frac{1}{m}e^{mx}$$

Corollary: Putting $m = 1,$ the above equation stands: $\int e^x \; dx \; = \; e^x$

In Calculus, it is sometimes necessary to add another operation with integration. More clearly, after deducing $\int f(x)dx = F(x)$, we may often require the value $F(a) - F(b)$; symbolically that is, $[F(x)]_b^a$, also written as (in a form with the derivative), $\int_b^a f(x)dx.$ Here, a & b both are constant, known as the upper & the lower limit respectively.

Moreover, $\int_b^a f(x)dx$ is called to be a definite integral since it has limits; whereas integrals of the kind $\int f(x)dx$, with which we dealt before, is called to be the indefinite ones.

Again, let $\int f(x)dx = F(x) + c,$ then——

$$\int_b^a f(x)dx = [F(x) + c]_b^a = \{F(a) + c\} - \{F(b) + c\} = F(a) - F(b);$$

which means definite integral does not depend on c and therefore, needs not to introduce it.

Example: Find out $\int_0^{\frac{\pi}{2}} \cos^3\theta \;\; d\theta.$

Solution: $\int_0^{\frac{\pi}{2}} \cos^3\theta \;\; d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta.\cos\theta \; d\theta = \int_0^{\frac{\pi}{2}} (1 - \sin^2\theta)\cos\theta \; d\theta$

Let, $y = \sin\theta$. Then, $\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta$; which means——

$$\frac{dy}{d\theta} = \cos\theta$$

that is, $\lim_{\Delta\theta \to 0} \frac{\Delta y}{\Delta\theta} = \cos\theta$

or, $\lim_{\Delta\theta \to 0} \Delta y \div \lim_{\Delta\theta \to 0} \Delta\theta = \cos\theta$

Since, y is a function of θ; hence, if $\Delta\theta \to 0$, then $\Delta y \to 0$

$$\therefore \lim_{\Delta y \to 0} \Delta y = \cos\theta . \lim_{\Delta\theta \to 0} \Delta\theta$$

So, we may write——

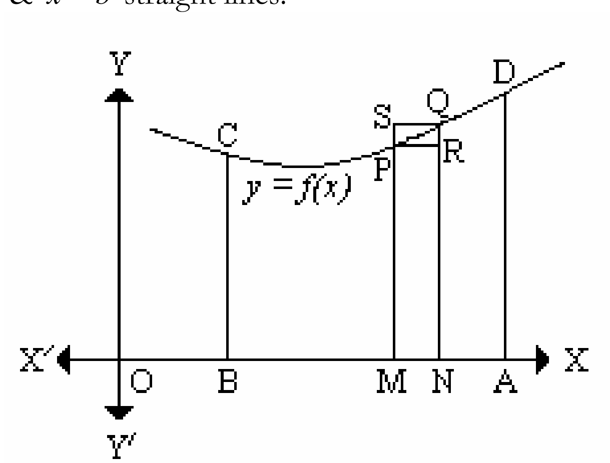
$$dy = \cos\theta \ \ d\theta$$

Now, when $\theta = \frac{\pi}{2}$, then $y = 1$ and when $\theta = 0$, then $y = 0$. Hence——

The required value $= \int_0^{\frac{\pi}{2}} (1 - \sin^2\theta)\cos\theta \ d\theta$

$= \int_0^1 (1 - y^2)dy = \int_0^1 1.dy - \int_0^1 y^2 dy \ = [y]_0^1 - \left[\frac{y^3}{3}\right]_0^1 \ = (1-0) - \left(\frac{1^3}{3} - \frac{0^3}{3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$.

To determine an area in a graph surrounded by a curve of a function $y = f(x)$, the X-axis and the $x = a$ & $x = b$ straight lines.



In the above figure, let CD be a curve of the function $y = f(x)$; AD & BC be straight lines of the equations $x = a$ & $x = b$ respectively. So, we have to determine the area of *ABCD*.

Now, let's take a point P (m,y) on the curve, very close to another one Q $(m+\Delta x, y+\Delta y)$. Again in the graph, let $PM \perp OX$, $QN \perp OX$, $PR \perp QN$ and $QS \perp MP$.

$$\therefore PM = y, \ QN = y + \Delta y, \ ,OM = m \text{ and } ON = m + \Delta x$$

$$\therefore MN = ON - OM = (m + \Delta x) - m = \Delta x$$

Now, let A be the area of *BMPC*. From the graph we see that, A is dependent on the position of the point M; that is, the length of OM; in other words, the value of m. Therefore, A is a function of m which can be defined as below:

$A = \phi(x)$ is said to be the area surrounded by *BC*, the curve $y = f(x)$, the X-axis and the straight line $x = x$.

Hence, $A = \phi(m)$ indicates the area surrounded by *BC*, the curve $y = f(x)$, the X-axis and the straight line $x = m$ (that is, *MP*).

$$\therefore \phi(m) = \text{The area of } BMPC$$

Similarly, putting $m + \Delta x$, a and b in the place of m, we obtain respectively——

$$\phi(m + \Delta x) = \text{The area of } BNQC$$

$$\phi(a) = \text{The area of } ABCD$$

$$\phi(b) = 0.$$

Now, The area of *PMNQ* $=$ The area of *BNQC* – The area of *BMPC*

$= \phi(m + \Delta x) - \phi(m)$

$= \Delta\phi(m)$ [Since, $\Delta F(x)$ and $F(x + \Delta x) - F(x)$ are the same.]

$= \Delta A$ [Since, $\phi(m) = A$]

Again, The area of *PMNR* $= y.\Delta x$

And The area of *SMNQ* $= (y + \Delta y).\Delta x$

From the figure it is clear that——

$$\text{The area of } PMNR < \text{The area of } PMNQ < \text{The area of } SMNQ$$

$$\therefore y.\Delta x < \Delta A < (y + \Delta y).\Delta x$$

$$\therefore y < \frac{\Delta A}{\Delta x} < (y + \Delta y)$$

$$\therefore \lim_{\Delta x \to 0} y < \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} < \lim_{\Delta x \to 0} (y + \Delta y)$$

Since, *y* is a function of *x*, hence, if $\Delta x \to 0$, then $\Delta y \to 0$.

$$\therefore \lim_{\Delta y \to 0} y < \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} < \lim_{\Delta y \to 0} (y + \Delta y)$$

$$\therefore y < \frac{dA}{dx} < y;$$

Which means: $$\frac{dA}{dx} = y$$

$$\therefore dA = y\ dx.$$

Integrating the above equation between the limits *a* & *b*, we get——

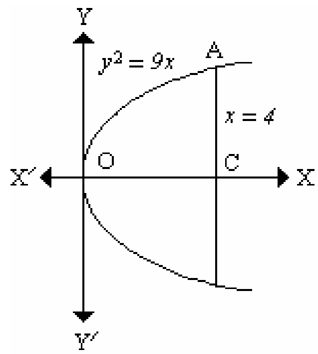
$$\int_b^a dA = \int_b^a y\ dx$$

But, $\int_b^a dA = [A]_b^a$ [Since, $\int dx = x$]

$= [\phi(m)]_b^a = \phi(a) - \phi(b) =$ The area of $ABCD - 0 =$ The required area

$\therefore$ The required area $= \int_b^a y\ dx = \int_b^a f(x)\ dx$

Example: Find out the area surrounded by the parabola $y^2 = 9x$ and the straight line $x = 4$.
Solution:



Since, $y^2 = 9x$; hence, $y = \sqrt{9x} = 3\sqrt{x}$

$\therefore$ The required area = 2 × The area of AOC

$$= 2\int_0^4 y\ dx$$

$$= 2\int_0^4 3\sqrt{x}\ dx$$

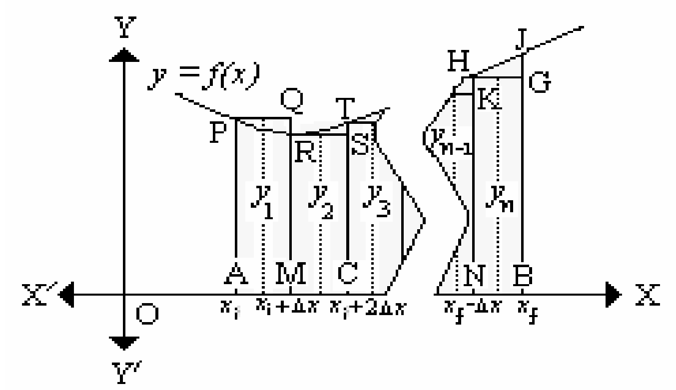
$$= 2.3\int_0^4 x^{\frac{1}{2}}\ dx$$

$$= 6\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= 6 \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$= 4 \times \left(4^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = 4 \times 4^{\frac{3}{2}} = 4 \times \sqrt{4^3} = 4 \times \sqrt{64} = 4 \times 8 = 32$$

- In the previous article, let's divide therein AB in n parts, each being Δx in length.



Now, if the number of parts; that is n, is very great; in other words, $\Delta x \to 0$, then the values of y lying between x_i & $x_i+\Delta x$ can be regarded as constant. Let y_1 be the value of $f(x_i)$.

Similarly, let y_2 be the constant value of y lying between the microscopic area fenced by $x_i+\Delta x$ & $x_i+2\Delta x$ and so on……to y_n between x_f & $x_f - \Delta x$. Now——

The area of $AMQP = y_1.\Delta x$

The area of $MCSR = y_2.\Delta x$

………………………

………………………

The area of $NBGH = y_n.\Delta x$

Adding all this together, we obtain——

The area of ($AMQP+MCSR+$......$+NBGH$) $= y_1.\Delta x + y_2.\Delta x + \ldots\ldots + y_n.\Delta x = \sum_{i=1}^{n} y_i.\Delta x$

$$\therefore \lim_{\Delta x \to 0} \text{[The area of } (AMQP+MCSR+......+NBGH)] = \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i.\Delta x$$

But, $\lim_{\Delta x \to 0}$ [The area of ($AMQP+MCSR+$......$+NBGH$)]

= The area of ($AMRP+MCTR+$......$+NBJH$)

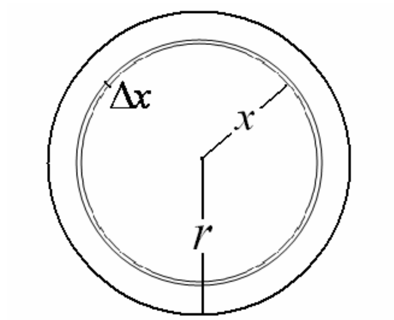
= The area of $ABJP$

= The area surrounded by the curve $y = f(x)$, the X-axis and the lines $x = x_f$ & $x = x_i$.

$= \int_{x_i}^{x_f} y \, dx$

$$\therefore \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i.\Delta x = \int_{x_i}^{x_f} y \, dx = \int_{x_i}^{x_f} f(x) \, dx$$

- Let's find out the area of a circle, radius of which is *r*.



At first, let the circle be divided in *n* rings, each having the same center as the original circle. Now, if we consider one of these rings having a radius x and a thickness Δx, we see that if it is cut and straightened like a string, its area ΔA_i can be regarded as that of a rectangle as long as the thickness, Δx is nearly zero. If it is not exactly zero, then we will not see an exact rectangle at all. Therefore using the idea of limits—

$$\Delta A_i = \lim_{\Delta x \to 0} y_i \Delta x$$

—where y_i is the circumference of the ring which equals to $2\pi x$.

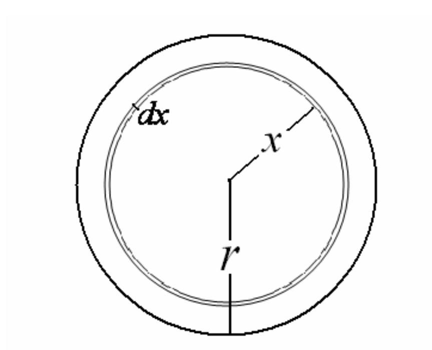
Now, we shall calculate the total area of the circle, A by adding up all the areas of the component rings which have their radii ranging from 0 to *r*.

$$\therefore A = \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} \lim_{\Delta x \to 0} y_i \Delta x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i \Delta x = \int_0^r y\,dx$$

$$= \int_0^r 2\pi x\,dx = 2\pi \int_0^r x\,dx = 2\pi \left[\frac{x^2}{2}\right]_0^r = 2\pi\left(\frac{r^2}{2} - \frac{0^2}{2}\right) = \pi r^2$$

$$\therefore \quad A = \pi r^2$$

The above procedure is often abbreviated as illustrated below—



Considering a ring of radius x and negligible thickness dx, its negligible area would be—

$$dA = 2\pi x dx$$

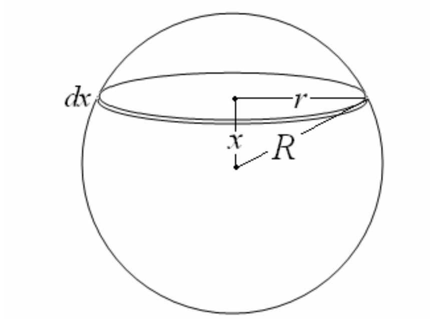
Now, integrating the above equation between $x = 0$ & $x = r$, we get—

$$\int dA = \int_0^r 2\pi x dx$$

$$\Rightarrow A = \int_0^r 2\pi x dx = ------ = \pi r^2$$

This method is much easier to handle than the previous one and it eventually does not alter the result a bit.

- Let's find out the volume of a sphere, radius of which is R.



The sphere can be chopped down into numerous discs. Let's consider such a disc which is at a distance x from the center of the sphere and has a radius r as well as a negligible thickness dx. Then the figure shows us that—

$$r^2 = R^2 - x^2$$

Again, the disc may be thought of a cylinder having r and dx as its radius and height. Then, its volume—

$$dV = \pi r^2 dx = \pi(R^2 - x^2)dx$$

Now, integrating the above equation between $x = 0$ & $x = R$ gives us the volume of a hemi-sphere—

$$\int dV = \int_0^R \pi(R^2 - x^2)dx$$

$$\therefore V = \int_0^R \pi R^2 dx - \int_0^R \pi x^2 dx$$

$$= \pi R^2 \int_0^R dx - \pi \int_0^R x^2 dx$$

$$= \pi R^2 [x]_0^R - \pi \left[\frac{x^3}{3}\right]_0^R$$

$$= \pi R^2 (R - 0) - \pi \left(\frac{R^3}{3} - \frac{0^3}{3}\right)$$

$$= \pi R^3 - \frac{1}{3}\pi R^3$$

$$= \frac{2}{3}\pi R^3$$

$\therefore$ Total volume of the complete sphere $= 2 \times \frac{2}{3}\pi R^3 = \frac{4}{3}\pi R^3$

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Thank you for reading this feature with so much patience. Any kind of question, complain or advice will be accepted gratefully.

Enjoy Physics.

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